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A MODIFIED PENALTY TERM FOR THE SEQUENTIAL UNCONSTRAINED MINIMIZATION TECHNIQUE FOR CONVEX PROGRAMMING PROBLEMS

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A MODIFIED PENALTY TERM FOR THE SEQUENTIAL UNCONSTRAINED MINIMIZATION TECHNIQUE FOR CONVEX PROGRAMMING PROBLEMS

by

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ABSTRACT

The Sequential Unconstrained Minimization Technique (SUMT) for Convex Programming Problems is modified by the introduction of an exponent in the penalty term. The exponent is introduced to increase the rate of convergence of the method for nonlinear problems with solutions on the boundary of one or more constraints. Convergence to the solution of the constrained problem is proved, and it is shown that SUMT is a special case of the general unconstrained function with the exponent equal to one. Results of a sample problem indicate that the rate of convergence is improved and that the computational time for solution is decreased for an exponent less than one.

TABLE OF CONTENTS

Section		Page
1.	Introduction	5
2.	The convex programming problem	6
3.	Sequential Unconstrained Minimization Technique (SUMT)	13
4.	Modification of the penalty function of SUMT	20
	4.1 Proof of convergence of the modified SUMT	22
5.	Behavior of the modified SUMT as a function of v	34
6.	Conclusions and Acknowledgements	40

LIST OF ILLUSTRATIONS

Figure		Page
1.a	Convex and concave functions	5
1.b	Global and Local maximums	6

LIST OF TABLES

I	Values of the P function for different values of v and r' .	35
II	Modified SUMT solutions of a sample problem	38

1. Introduction.

Since 1951, when Kuhn and Tucker extended the method of Lagrange multipliers to include inequality constraints, the techniques used for optimization of nonlinear problems have developed rapidly. The growth of this mathematical tool in the last decade is due to its successful application to many Military and Industrial problems. Section 2 describes the general convex programming problem and the theorems and conditions that assure convergence. Section 3 outlines one iterative method, the so called Sequential Unconstrained Minimization Technique, hereafter called SUMT.

SUMT converts a constrained convex programming problem, i.e., minimize $f(x)$, subject to $g_i(x) \geq 0$, $i = 1,2,\ldots m$, to an unconstrained problem; minimize $f(x) + r \sum_{i=1}^{m} 1/g_i(x)$, $r > 0$. A second order gradient method is used to minimize the unconstrained function for a fixed value of r, then r is reduced and the procedure is repeated, until the solution of the constrained $f(x)$ is approximated.

In section 4, the unconstrained problem of SUMT is modified to the form $f(x) + r \sum_{i=1}^{m} \left(1/g_i(x)\right)^{v}$, $r > 0$, $v > 0$. The proof of convergence of the modified SUMT is in section 4.1. SUMT was modified to increase the rate of convergence for convex programming problems with solutions on the boundary of one or more constraints.

The effect of the parameter v on the unconstrained problem and on its gradient is analyzed in section 5. The increase in the rate of convergence of SUMT is shown for a sample problem.

2. The Convex Programming Problem.

The problem of optimizing a function, subject to constraints, occurs frequently in industry, economics, and in pure and applied mathematics. The development of the high speed, digital computer in the late forties of this century, has generated a new interest in optimization problems that were too complicated or time consuming for hand computation.

In the general case we wish to solve a problem of the form; find an n-dimensional rector $\mathbf{x} = (x_1, x_2, \ldots x_n)$ that maximizes or minimizes the objective function $f(x)$, subject to the constraints $g_i(x) \geq 0$. Additional constraints such as, all $x_i \geq 0$, or combinations thereof may be required and in this paper will be considered absorbed into the $g_i(x)$'s.

In 1947 George Dantzig[10] devised the simplex algorithm for solving the general linear programming problem. That is, a problem of the form, optimize $f(x) = \sum_{j=1}^{n} c_j x_j$, subject to $g_i(x) = \sum_{j=1}^{m} a_{ij} x_j$,i=1,...m, where c_j and a_{ij} are known constants. The simplex algorithm is capable of solving linear problems with several hundred variables and/or constraints. The majority of the practical problems solved by the simplex method are linear approximations of nonlinear ones, and considerable effort has been expended to find a direct method of solving nonlinear programming problems.

To date no general method has been found for nonlinear programming; however, many special methods exist for solving particular types of nonlinear problems. The programmer, faced with a nonlinear optimization problem, must decide, based on his knowledge of the functions and the accuracy desired, whether to use a linear approximation and the simplex method or try to find a nonlinear method which solves his problem.

The method of Lagrange multipliers provides the classical approach to optimization problems.[10] Let f(x) be the objective function to be optimized, subject to $g_i(x) = 0$, i = 1,2,...m, then form the function:

$$L(x, \lambda) = f(x) + \lambda_1 g_1(x) + \lambda_2 g_2(x) + \cdots + \lambda_m g_m(x)$$

where λ_i are constants,

then differentiate $L(x, \lambda)$ with respect to x_i and λ_i and set each equal to zero:

$$\frac{\partial L(x,\lambda)}{\partial x_1} = 0 \qquad \frac{\partial L(x,\lambda)}{\partial \lambda_1} = 0$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial L(x,\lambda)}{\partial x_i} = 0 \qquad \frac{\partial L(x,\lambda)}{\partial \lambda_i} = 0$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial L(x,\lambda)}{\partial x_n} = 0 \qquad \frac{\partial L(x,\lambda)}{\partial \lambda_m} = 0$$

Lagrange discovered that if a vector x is a solution to the optimization problem it will also satisfy the above n + m equations. The λ_i's are known as the Lagrange multipliers and are interpreted in economic problems as the "shadow prices"[10]. The method of Lagrange is of great theoretical value but unfortunately is not very useful in practice and all but very simple problems can be solved easier by other methods.

In 1951 Kuhn and Tucker[11] generalized the theory of Lagrance multipliers to inequality constraints and non-negative variables. Before discussing the Kuhn-Tucker theorem it is necessary to provide some definitions:

> Convex Function[10]: The function f(x) is said to be convex over a convex set X in E^n if for any two points x_1 and x_2 in X and for all λ, $0 \leq \lambda \leq 1$.
>
> $$f[\lambda x_2 + (1-\lambda)x_1] \leq \lambda f(x_2) + (1-\lambda) f(x_1)$$

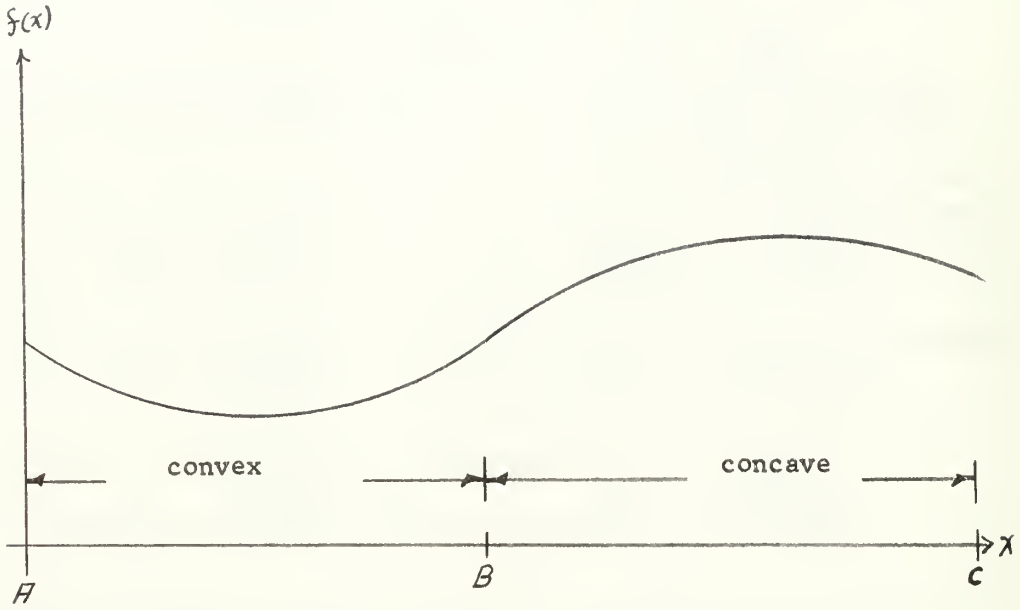

Figure Ia

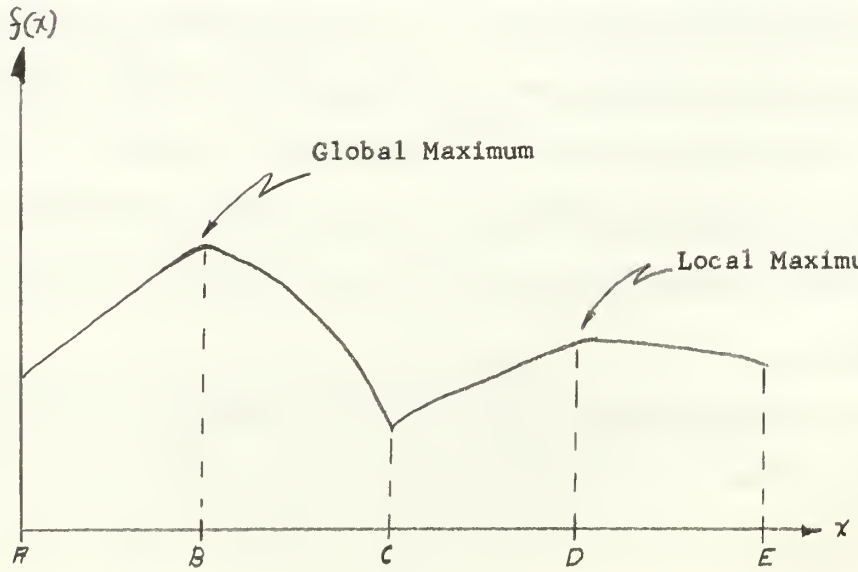

Figure Ib

Concave Function [10]: The function f(x) is said to be concave over a convex set in X in E^n, if for any two points x_1 and x_2 in X, and for all λ, $0 \leq \lambda \leq 1$.

$$f[\lambda x_2 + (1-\lambda)x_1] \geq \lambda f(x_2) + (1-\lambda) f(x_1)$$

The function shown in figure Ia is convex over the interval $A \leq x \leq B$, and concave over the interval $B \leq x \leq C$; however, it is neither convex nor concave over the interval $A \leq x \leq C$.

An equilivalent definition of convex function is: if f(x) is twice continuously differentiable then the matrix of second partials of f(x), i.e., $\left\| \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right\|$, is positive semi-definite, (negative semi-definite for concave functions).

Note: if f(x) is convex, then -f(x) is concave.

Strictly Convex (concave): A function f(x) is strictly convex (concave) if only the inequality in the above definition holds.

It is obvious from the definitions that,

a) a linear function is both concave and convex,

b) the sum of concave (convex) functions is a concave (convex) function.

Global Maximum: The function f(x) defined over a closed set X in E^n is said to take on a global maximum over X at the point x* if $f(x) \leq f(x^*)$ for every point $x \in X$.

Local Maximum: The function f(x), defined at all points in a δ -neighborhood of x* in E^n, is said to take on a local maximum at x*, if for all x in the δ -neighborhood, i.e., $|x^*-x| < \delta$, $f(x) \leq f(x^*)$.

The definitions of a global minimum and local minimum are obtained by reversing the inequalities. The function shown in Figure Ib has, for $A \leq x \leq E$, a global maximum at B and a local maximum at D, two local minima at A and E, and a global minimum at C.

The convex programming problem can now be stated:

Problem A.

Minimize a continuously differentable convex function

$f(x)$

Subject to the constraints

$$g_i(x) \geq 0 \qquad i = 1,2,\ldots m,$$

where each constraint is continuously differentable and concave.

The desirable feature of the above restrictions is that a local minimum is also a global minimum.

Problem B.

Form the Lagragian function

$$Q(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

then find the vectors $\bar{x}$ and $\bar{\lambda}$ such that:

$$Q(x,\bar{\lambda}) \geq Q(\bar{x},\bar{\lambda}) \geq Q(\bar{x},\lambda)$$

for all $x \geq 0$, $\lambda \geq 0$

i.e., $x_i \geq 0$, $\lambda_i \geq 0$, where x_i and λ_i are the components of the vectors x and λ .

Kuhn and Tucker[11] established the necessary and sufficient conditions for $\bar{x}$ and $\bar{\lambda}$ to provide a solution to B.

Let $\Phi_x = \left[\frac{\partial Q}{\partial x_i}\right]_{(\bar{x},\bar{\lambda})}$ and $\Phi_\lambda = \left[\frac{\partial Q}{\partial \lambda_i}\right]_{(\bar{x},\bar{\lambda})}$

the necessary conditions take the form

$$\Phi_x \geq 0, \qquad \Phi_x \circ \bar{x} = 0 \qquad \bar{x} \geq 0$$

where the second equation is the dot product of two vectors.

Together with the above conditions,

$$Q(x,\bar{\lambda}) \leq Q(\bar{x},\bar{\lambda}) + \Phi_x \circ (x - \bar{x})$$

$$Q(\bar{x},\lambda) \geq Q(\bar{x},\bar{\lambda}) + \Phi_\lambda \circ (\lambda - \bar{\lambda})$$

$$x \geq 0 \ , \ \lambda \geq 0$$

form the sufficient conditions for the solution of problem B.

The Kuhn and Tucker "Equilvalance Theorem", which states that

problem A is equilvalent to problem B, has been the basis for important additional theoretical work in nonlinear programming, such as, "differential" gradient methods of Arrow, Hurwicz, and Uzawa[2], and the Duality Theorems.

If problem A has a strictly convex objective function and there exist a point x in the feasible domain such that for each i

$$g_i(x) > 0.$$

then the dual problem to problem A can be stated[8].

Problem C.

Maximize $G(x,\lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$

Subject to $\nabla_x f(x) = \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x)$, $\lambda_i \geq 0$

where ∇_x is the gradient of the function with respect to the vector x. If either problem A or C has a finite solution the other does and moreover Minimum f(x) = Maximum G($\bar{x}$, $\bar{\lambda}$), and if $\bar{x}$ is a solution to problem A then $\bar{x}$ together with $\bar{\lambda}$ is a solution to C. The Dual or complimentary variables can be interpreted with properties of the system, for example, if the original variables are such that their magnitudes increase proportionally to the system (mass, cost, etc.) then the dual variables magnitudes will be independent of the size of the system (pressure, price, etc.) and conversely.[6]

Hadley[10], Dorn[6] and Arrow[2] are excellent references for the various theoretical and computational methods in current use to solve nonlinear problems. The methods will not be covered in this paper; however, it should be pointed out that each method has advantages and disadvantages for a particular type of problem which should be studied

carefully prior to use.

The field of nonlinear programming is young. Most of the theoretical and computational work is less than 10 years old, and considerable work remains to be done. The age of the field shows in the lack of literature on this subject. Fiacco and McCormack[8] sum up the situation very well in the following quotation:

> We have already deplored the general dearth of such information in the literature on nonlinear programming, which not only makes comparative analysis between practitioners exceedingly difficult, but makes it virtually impossible for a potential user to decide whether a given problem can be solved and, if it can, to estimate the required effort.

3. Sequential Unconstrained Minimization Technique.

In 1961, Carroll[4] proposed that the constrained, concave programming problem; maximize f(x), subject to $g_i(x) \geq 0$, i = 1,2,...m, x is an n-dimensional vector, be modified to the unconstrained problem:

$$\text{Maximize} \quad P(x, r) = f(x) - r\sum_{i=1}^{m} \frac{w_i}{g_i(x)}$$

$$\text{with} \quad w_i \geq 0 \text{ and } r > 0$$

Carroll's unconstrained P function has several desirable properties;

a) the summation term (called the penalty function) approaches $-\infty$ if the boundary of a constraint is reached;

b) if the functions f(x) and $g_i(x)$ have continuous first and second partial derivatives inside the feasible region then the well known necessary conditions for an unconstrained function to have a local maximum applies, that is; the gradient vanishes at that point and the matrix of second partials is negative semi-definite[1];

c) if f(x) is strictly concave or any $g_i(x)$ is strictly concave then the local maximum is the global maximum;

d) first or second order gradient methods may be used to maximize P(x, r);

e) when a maximum of P(x, r) is reached for a fixed value of r, r can be reduced ($r_1 > r_2 > \cdots r_k \cdots > 0$) and the new P(x, r) solved for a maximum;

f) as $r_k \rightarrow 0$ the maximum of $P(x, r_k)$ approaches the maximum of f(x) and the penalty term approaches zero.

Carroll demonstrated that the computational method will solve convex programming problems.

Fiacco and McCormick[8] proved Carroll's conjecture for the corresponding convex programming problem. Their work established the proof of the following:

Given the convex programming problem (Problem A)

Minimize $f(x)$

Subject to $g_i(x) \geq 0$ $i = 1,2,\ldots m$

(where x is a n-dimensional vector)

Define the function:

$$P(x, r_k) = f(x) + r_k \sum_{j=1}^{m} \frac{1}{g_i(x)}$$

where $r_1 > r_2 > \cdots > r_k > \cdots 0$

Define $x(k)$ as the vector minimizing $P(x, r_k)$, let V_0 be the constrained minimum value of $f(x)$.

Then:

$$\lim_{\substack{k \to \infty \\ r_k \to 0}} P(x, r_k) = V_0$$

The following additional conditions must hold.

C1: $R^\circ = \{x \mid g_i(x) > 0, i = 1, \cdots, m\}$ is not empty, denote by R the closure of R°

C2: $f(x)$ and $-g_i(x)$ are convex and twice continuously differentiable for $x \in R$.

C3: For every finite h, $\{x \mid f(x) \leq h, x \in R\}$ is a bounded set.

C4: For every $r > 0$, $P(x, r)$ is strictly convex.

Condition C1 is necessary for the method to apply since it is an inside method, condition C3 ensures that a local minimum is achieved at a finite point, and condition C4 is necessary to ensure that a local minimum is the global minimum. Condition C4 is satisfied if any of the following statements apply to the problem;

a) f(x) is strictly convex,

b) there are n independent linear constraints on the problem (the constraints $\chi_i \geq 0$ i = 1,...n are a special case).

c) any constraint $g_i(x)$ is strictly concave.

Fiacco and McCormack[8] showed that the manner in which the primal problem is solved yields a set of points that are dual-feasible, problem C, and approach the dual optimum in the limit as $r_p \longrightarrow 0$. They further demonstrated that the solution to the primal and dual problems for a fixed value of r, say r_p, bound the final solution to the problem, that is: if x (p) is the optimum solution to P(x, r_p) and V_0 is the optimum solution of the constrained f(x) then:

$$G[\chi(p), \lambda] \leq V_0 \leq f[\chi(p)] \leq P[\chi(p), r_p]$$

This theoretical development provides a criterion for termination of the computational method. Fiacco and McCormick also proved that the optimum solution to the subproblem; (minimize P(x, r_p)), successively decreases to V_0 ; that is, for $r_p > r_{p+1} > \cdots > 0$ then

$$P[\chi(p), r_p] > P[\chi(p+1), r_{p+1}]$$

Fiacco and McCormick with Mylander[9] developed a computer routine to apply the Sequential Unconstrained Minimization Technique (SUMT) to convex programming problems, using a modification to Davidon's[5] second order gradient method. SUMT program written in FORTRAN-4 machine language is available from the SHARE library under distribution number SDA 3189. The program is still in the experimental stage and the authors are in the process of rewriting it to increase its' efficiency and accuracy.

The program as now written will handle up to 100 variables and 200 constraints (including any restrictions on the variables). SUMT has

solved several nonconvex problems; however, since there is no theoretical justification to the convergence of a non-convex problem the user must interpret the answers carefully.

An example might serve to illustrate the principles of SUMT.

$$\begin{aligned} &\text{Minimize} && x_1 + x_2 \\ &\text{Subject to} && x_1 - 1 \geq 0 \\ & && x_2 - 1 \geq 0 \end{aligned}$$

Both the objective function and the constraints are linear and therefore are both convex and concave. The four conditions required by SUMT are met.

Using the method of Lagrange:

$$L(x,\lambda) = x_1 + x_2 + \lambda_1(x_1-1) + \lambda_2(x_2-1)$$

$$\partial L/\partial x_1 = 1 + \lambda_1 = 0 \Rightarrow \lambda_1 = -1$$

$$\partial L/\partial x_2 = 1 + \lambda_2 = 0 \Rightarrow \lambda_2 = -1$$

$$\partial L/\partial \lambda_1 = x_1 - 1 = 0 \Rightarrow x_1 = 1$$

$$\partial L/\partial \lambda_2 = x_2 - 1 = 0 \Rightarrow x_2 = 1$$

at the point (1,1), f(x) = 2 the actual minimum of the constrained problem.

Using SUMT:

$$P(x,r) = x_1 + x_2 + \frac{r}{x_1-1} + \frac{r}{x_2-1}$$

the necessary condition for an unconstrained minima;

$$\partial P/\partial x_1 = 1 - \frac{r}{(x_1-1)^2} = 0 \Rightarrow x_1 = 1 \pm \sqrt{r}$$

$$\partial P/\partial x_2 = 1 - \frac{r}{(x_2-1)^2} = 0 \quad \Rightarrow \quad x_2 = 1 \pm \sqrt{r}$$

Since $r > 0$ and $1-\sqrt{r}$ is not a feasible point, it is therefore rejected as a possible solution, then

$$x_1 = 1 + \sqrt{r} \qquad x_2 = 1 + \sqrt{r}$$

then:

$$\bar{x}_1 = \lim_{r \to 0} x_1(r) = \lim_{r \to 0} 1 + \sqrt{r} = 1$$

$$\bar{x}_2 = \lim_{r \to 0} x_2(r) = \lim_{r \to 0} 1 + \sqrt{r} = 1$$

$$f(\bar{x}) = \lim_{r \to 0} f(x(r)) = \lim_{r \to 0} 2 + 2\sqrt{r} = 2$$

The same minimum was achieved using the method of Lagrange. To demonstrate the iterative procedure,

let $r_1 = 1$:

$$P(x, 1) = x_1 + x_2 + \frac{1}{x_1 - 1} + \frac{1}{x_2 - 1}$$

$$x_1 = 1 + \sqrt{1} = 2 \qquad x_2 = 1 + \sqrt{1} = 2$$

and the minimum of $P[x(1), 1] = 6$

Let $r_2 = \frac{1}{4}$:

$$P(x, 1/4) = x_1 + x_2 + \frac{1}{4(x_1 - 1)} + \frac{1}{4(x_2 - 1)}$$

$$x_1 = 1 + 1/4 = 3/2 \qquad x_2 = 1 + 1/4 = 3/2$$

and the minimum of $P[x(2), 1/4] = 29/9$

and $P[x(2), 1/4] < P[x(1), 1]$.

Fiacco[7,8] further showed that a modification to the unconstrained method could be used to find a feasible starting point if one was not known. Given a starting point x^0 with at least one of the constraints not satisfied.

We define index sets

$$S = \{ s \mid g_s(x) \leq 0 \},$$

$$T = \{ t \mid g_t(x) > 0 \}.$$

Pick an element of S say s_1. We then proceed to minimize

$$P(x, r) = -g_{s_1}(x) + r \sum_{t \in T} \frac{1}{g_t(x)}$$

At each point during the minimization process the unsatisfied constraints are checked and in case $g_s(x) > 0$, $s \in S$, then s is shifted into T. The process is continued until a value of x is found for which $g_{s_1}(x) > 0$. The index s_1 is shifted into T and if S is not empty another element of S is taken and the procedure is repeated until S is empty.

SUMT has several desirable properties for solving nonlinear convex programming problems, they are:

a) If a feasible starting point is not known the method can determine one.

b) The method will determine a minimum if it is an interior point or if it lies on the boundary of a constraint.

c) The amount of computer time needed to find a minima is compatible with other methods in current use.

d) The solution to the dual and the Lagrange multipliers yield desirable additional information.

SUMT however is not without disadvantages, they are:

a) In highly nonlinear problems it is frequently difficult to determine if the objective function (or the constraints) is convex

(concave) in the region of interest, and also if the interior is a connected region. This is true of all nonlinear methods.

b) If the solution lies on a boundary it will be impossible to get an exact solution, and a high price (computer time) will be paid if it is desired to get very close to the solution.

c) The initial value of r and a method of reducing r, at each subproblem minimization, has not been thoroughly investigated, although a method has been found that works reasonably well in practice.

This paper will investigate a modified penalty function, which preserves the desirable features of SUMT and improves the undesirably slow convergence.

4. Modification of the penalty function of SUMT

The idea of converting a constrained function to an unconstrained problem whose solution approaches the constrained solution is not new. The conditions imposed on the constrained problem provide an unconstrained function that can be minimized by existing methods. Can SUMT be modified to increase its rate of convergence without effecting its desirable features? What properties must an unconstrained function have in order that it converge to the minimum of the constrained problem?

Intuitively the unconstrained problem should remain convex, in order that we may use calculus methods to find the minimum. The unconstrained problem must have some "built-in" method of remaining inside the feasible region and it should be monotonically related to the objective function and the constraints. If an iterative procedure is to be used for solving the unconstrained problem, then each iteration should yield a point that successively minimizes the objective function, (that is, $f(x(p)) \geq f(x(p+1))$ for p integer > 0). And most important, the minimum of the sequence of unconstrained functions must equal the minimum of the constrained objective function.

SUMT was modified, by the author of this paper, for FORTRAN-63 and the CDC 1604 computer of the U. S. Naval Postgraduate School. The accuracy of the conversion was tested using the sample problem furnished with the SHARE library program[9]. Several linear and non-linear convex programming problems with mimima occurring on the boundary of one or more constraints were tested. In all cases SUMT moved close to the solution expeditiously; however, once close to a boundary, movement in the direction of mimima slowed and excessive computer time was required to achieve the desired accuracy.

For example, in the following problem:

$$\text{Minimize} \quad f(x) = x_1^2 + (x_2 - 4)^2$$

Subject to:

1. $2x_1 + x_2 - 6 \geq 0$
2. $x_1 - 1 \geq 0$
3. $x_2 \geq 0$

The solution is at (1,4) and Min f(x) = 1.

Subproblem Number	x_1	x_2	f[x(p)]	$P(x(p), r_p)$	r_p	Cumulative Computer time, Seconds*
0	2.0	2.1	-	-	-	0
1	1.37	3.38	2.27	17.5	1.37	-
2	1.219	3.80	1.52	4.97	.34	.716
3	1.167	4.084	1.37	2.206	.086	1.583
4	1.109	4.100	1.24	1.53	.021	2.7
5	1.054	4.09	1.11	1.25	5.3×10^{-3}	3.9
6	1.02	4.07	1.05	1.12	1.3×10^{-3}	5.3
7	1.012	4.059	1.02	1.06	3.3×10^{-4}	6.566
8	1.006	4.04	1.014	1.029	8.4×10^{-5}	8.050
.	.	.	.	.	.	.
.	.	.	.	.	.	.
.	.	.	.	.	.	.
16	1.0000	4.001	1.00	1.0001	1.2×10^{-9}	20.433

*Not including compiler time.

It is obvious that we are close to the solution at the sixth subproblem minimum. At that time the values of the constraints are: $g_1(x) = .096$, $g_2(x) = .02$, and $g_3(x) = 4.07$. The first and second constraints are binding and in the direction of mimima they will become even smaller causing an increase in the P function that nullifies the decrease in r. The rate of movement towards the mimima is reduced as the boundary is

approached.

One obvious way to increase the rate of convergence of SUMT is to modify the penalty term so that it's rate of increase, as a boundary is approached, is reduced. The modification must not change any of the desired features of SUMT. The following P function retains all the necessary properties of SUMT, and for $v < 1$, buffers the rate of increase of the penalty term.

$$P[x, r, v] = f(x) + r\sum_{i=1}^{m}\left(\frac{1}{g_i(x)}\right)^v$$

for $0 < v < 1$, $r > 0$.

It will be shown that P(x, r, v) has all the properties of P(x,r) and that P(x,r) is but a special case of P(x, r, v), that is with v = 1. P(x, r, v), with $v < 1$, buffers the rate of increase of the binding constraints as the following table demonstrates:

$g_i(x)$	$(1/g_i(x))^v$	v = 1	v = .5	v = .25	v = .1
1	1	1	1	1	1
.5	2	2	1.41	1.32	1.07
.25	4	4	2	1.74	1.15
.1	10	10	3.16	2.51	1.26

4.1 We restate the convex nonlinear programming problem in its new form:

Minimize f(x)

Subject to $g_i(x) \geq 0$ i = 1,2,...m

Define the function:

$$P[x, r_k, v] = f(x) + r_k\sum_{i=1}^{m}\left(\frac{1}{g_i(x)}\right)^v$$

In addition the following conditions hold:

C1: $R^\circ = \{ x \mid g_i(x) > 0,\ i = 1,2,...m \}$ is not empty. Denote by R the closure of R°

C2: $f(x)$ and $-g_i(x)$ are convex and twice continuously differentiable for $x \in R$.

C3: for every finite h, $\{ x \mid f(x) < h, x \in R \}$ is a bounded set.

C4: for every $r > 0$ and $v > 0$, $P[x,r,v]$ is strictly convex.

C5: $r_1 > r_2 > \ldots r_p > \ldots\ldots 0$

Define: $x(p)$ as the point that minimizes $P(x, r_p, v)$.

Lemma 1

If $g(x)$ is concave then $r[1/g(x)]^v$ is convex for $v > 0$, $r > 0$, and $x \in R^\circ$.

Proof:

The matrix of second partials of $r[1/g(x)]^v$ is:

$$\left\| \frac{\partial^2 r\left(\frac{1}{g(x)}\right)^v}{\partial x_i \partial x_j} \right\| = \left\| \frac{-rv}{[g(x)]^{v+1}} \cdot \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \right\| + \left\| \frac{v(v+1)r}{(g(x))^{v+2}} \cdot \frac{\partial g(x)}{\partial x_i} \cdot \frac{\partial g(x)}{\partial x_j} \right\|$$

for $x \in R^\circ$, $g(x) > 0$, $r > 0$, and $v > 0$, then the terms,

$$\frac{rv}{[g(x)]^{v+1}} \quad \text{and} \quad \frac{v(v+1)r}{[g(x)]^{v+2}} \quad \text{are positive}$$

and can be factored out of their respective matrices on the right.

$$\left\| \frac{\partial^2 r\left(\frac{1}{g(x)}\right)^v}{\partial x_i \partial x_j} \right\| = \frac{rv}{[g(x)]^{v+1}} \left\| \frac{-\partial^2 g(x)}{\partial x_i \partial x_j} \right\| + \frac{v(v+1)r}{[g(x)]^{v+2}} \left\| \frac{\partial g(x)}{\partial x_i} \cdot \frac{\partial g(x)}{\partial x_j} \right\|$$

since $g(x)$ is concave by condition C2,

$$\left\| \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \right\| \quad \text{is negative semi-definite}$$

therefore $$\left\| \frac{-\partial^2 g(x)}{\partial g(x)} \right\| \quad \text{is positive semi-definite.}$$

The second matrix on the right is of rank 1, or less, (the rank of a matrix is defined as the order of the largest non-vanishing determinant), since all determinants of order 2 or greater are equal to zero.

Proof:

$$\begin{vmatrix} \frac{\partial g(x)}{\partial x_i} \cdot \frac{\partial g(x)}{\partial x_j} & \frac{\partial g(x)}{\partial x_i} \cdot \frac{\partial g(x)}{\partial x_k} \\ \frac{\partial g(x)}{\partial x_l} \cdot \frac{\partial g(x)}{\partial x_j} & \frac{\partial g(x)}{\partial x_l} \cdot \frac{\partial g(x)}{\partial x_k} \end{vmatrix} = 0 \text{ for all i, j, k, l.}$$

Multiplying on the left by Y^T and on the right by Y, where Y is an arbitrary n-dimensional vector not equal to the null vector, yields the following Quadratic form:[3]

$$Y^T \left\| \frac{\partial^2 r\left(\frac{1}{g(x)}\right)^r}{\partial x_i \partial x_j} \right\| Y = \frac{r^r}{[g(x)]^{r+1}} Y^T \left\| \frac{-\partial^2 g(x)}{\partial x_i \partial x_j} \right\| Y$$

$$+ \frac{r(r+1)r}{[g(x)]^{r+2}} Y^T \left\| \frac{\partial g(x)}{\partial x_i} \cdot \frac{\partial g(x)}{\partial x_j} \right\| Y$$

$$\geq 0$$

For all values of the vector Y , except $Y = 0$, the first matrix on the right is positive semi-definite and therefore it's quadratic form is greater than or equal to zero for all values of Y . The second matrix on the right is of rank 1, hence the quadratic form is factorable. Browne[3] (pages 111-112) proves that the quadratic form

$$\sum_i \sum_j a_{ij} y_i y_j = \left(\sum_i c_i y_i\right)\left(\sum_j d_j y_j\right)$$

if the rank of $\|a_{ij}\|$ is one.

Therefore

$$\frac{v(v+1)r}{[g(x)]^{v+2}} Y^T \left\| \frac{\partial g(x)}{\partial x_i} \cdot \frac{\partial g(x)}{\partial x_j} \right\| Y = \frac{v(v+1)r}{[g(x)]^{v+2}} \sum_i \sum_j \frac{\partial g(x)}{\partial x_i} \frac{\partial g(x)}{\partial x_j} y_i y_j$$

$$= \frac{v(v+1)r}{[g(x)]^{v+2}} \left(\sum_i \frac{\partial g(x)}{\partial x_i} y_i\right)\left(\sum_j \frac{\partial g(x)}{\partial x_j} y_j\right)$$

$$= \frac{v(v+1)r}{[g(x)]^{v+2}} \left(\sum_i \frac{\partial g(x)}{\partial x_i} y_i\right)^2$$

$$\geq 0$$

For all values of Y .

Hence by definition $r[1/g(x)]^v$ is convex. This completes the proof of Lemma 1.

The sum of convex functions is a convex function, hence P(x, r, v) is convex.

At this point another condition is imposed on the basic non-linear convex programming problem.

Condition C6: The greatest lower bound of f(x), $x \in R$

is finite i.e., $f(x) \geq V_0 > -\infty$.

Lemma 2: Under conditions C1 thru C6 P(x, r, v) is bounded below for $x \in R^\circ$ and any $r > 0$, $v > 0$.

Proof:

$$P[x, r, v] = f(x) + r\sum_{i=1}^{m}\left(\frac{1}{g_i(x)}\right)^v$$

$$\geq \min_{x \in R^\circ} f(x) \geq V_0 > -\infty$$

($R^\circ \subset R$, condition C1 and C6)

Define: x° as the interior point at which minimization begins.

Lemma 3: a) Any local minimum of P(x, r, v) is in R° and is finite.

b) At least one such point exists.

Proof: Condition C1, (R° is not empty) is sufficient for the existence of a point x°. Let r° be any value of $r > 0$.

Define $M_0 = P(x^\circ, r^\circ, v) > -\infty$

(Lemma 2 and $v > 0$)

For any boundary point x^B, $g_i(x^B) = 0$ for some i, (condition C2) hence the P function is not defined at x^B and any local minimum must be an element of R°.

It is then possible to form the sets

$$S_0 = \{x \mid f(x) \leq M_0, x \in R\}$$

and

$$S_i = \{x \mid r^v \sum \left(\frac{1}{g_i(x)}\right)^v \leq M_0 - V_0, x \in R\}.$$

$$i = 1, \cdots m$$

Note: S_0 and S_i, i = 1,...m are closed.

Let

$$S = \bigcap_{i=0}^{m} S_i$$

For any point $y \in R$ and $y \notin S$, either

$$f(y) > m_o \quad \text{or} \quad r^o \sum_i \left(\frac{1}{g_i(y)}\right)^\nu > m_o - V_o \qquad \text{(for some i)}$$

If $f(y) > m_o$ then;

$$P[y, r^o, \nu] = f(y) + r^o \sum_i \left(\frac{1}{g_i(y)}\right)^\nu > f(y) > m_o$$

If $r^o \sum_i \left(\frac{1}{g_i(y)}\right)^\nu > m_o - V_o$ then,

$$P[y, r^o, \nu] = f(y) + r^o \sum_i \left(\frac{1}{g_i(y)}\right)^\nu$$

$$> V_o + (m_o - V_o) = m_o$$

By the definition of a local minimum, any local minimum of P(x, r°, v) must be in S, (if it exists). By construction S is non-empty ($x^o \in S$), and is closed and bounded, (condition C3 insures that S is bounded). Hence, part a is proved.

P(x, r°, v) is continuous on a compact set S and hence assumes a global minimum in S. This implies the existence of a local minimum in R. This proves part b.

Theorem 1: Subject to conditions C1-C6 the function P(x,r,v) has at least one local minimum $x(r) \in R^o$.

Furthermore:

a) x(r) is finite

b) $\nabla_x P[x(r), r, \nu] = 0$

c) $\left\| \dfrac{\partial^2 P[x(r), r, \nu]}{\partial x_i \, \partial x_j} \right\|$ is a positive definite matrix.

Proof:

a) $x(r)$ is finite element of R° by Lemma 3.

b) $P(x(r), r, v) = 0$ is the necessary condition for a minimum at $x(r)$.

c) the matrix of second partials is positive definite throughout the feasible region by definition of a strictly convex function.

Lemma 4. There is at most one local minimum of $P(x, r, v)$ for any $r > 0$, and a fixed value of $v > 0$.

Proof by contradiction: Assume there exist two points $x^1(r)$ and $x^2(r)$ for which $P(x, r, v)$ has a local minimum value in R. By condition C4, $P(x, r, v)$ is strictly convex, then:

$$P[[(1-\lambda)x^1(r) + \lambda x^2(r)], r, v] < (1-\lambda)P[x^1(r), r, v] + \lambda P[x^2(r), r, v]$$

Since $x^1(r)$ is assumed to be a local minimum

$$P[x^1(r), r, v] \leq P[(1-\lambda)x^1(r) + \lambda x^2(r), r, v]$$

$$< (1-\lambda)P[x^1(r), r, v] + \lambda P[x^2(r), r, v]$$

Transposing and collecting terms

(1) $$\lambda P[x^1(r), r, v] < \lambda P[x^2(r), r, v]$$

Using the same procedure with $(1-\lambda)x^1(r) + \lambda x^2(r)$ yields

(2) $$\lambda P[x^2(r), r, v] < \lambda P[x^1(r), r, v]$$

clearly both equations 1 and 2 are impossible.

Now consider the iterative method of minimizing the $P(x, r, v)$

function. Starting from an interior point x° and with a fixed value of $v > 0$ and a value of $r > 0$, minimize $P(x, r_1, v)$. Such a minimum exists by theorem 1 and is unique by Lemma 4. Then reduce r, i.e., $r_2 < r_1$ and minimize $P(x, r_2, v)$ using the point $x(1)$, (minimum of $P(x, r_1, v)$, as a starting point, etc. The final steps of the proof will be to show that such an iterative method converges to the solution of the initial non-linear convex programming problem.

Lemma 5. For $r_k > r_{(k+1)} > 0$,

$$P[x(k), r_k, v] > P[x(k+1), r_{k+1}, v]$$

Proof:

$$P[x(k), r_k, v] = f[x(k)] + r_k \sum_i \left(\frac{1}{g_i[x(k)]}\right)^v$$

$$> f(x(k)) + r_{k+1} \sum_i \left(\frac{1}{g_i[x(k)]}\right)^v$$

$$r_k > r_{k+1}, \quad x(k) \in R^\circ$$

$$\geq P[x(k+1), r_{k+1}, v]$$

since the point $x(k)$ is the starting point for minimization of $P(x, r_{k+1}, v)$.

Theorem 2: Basic convergence theorem

Under conditions C1 to C6, the values $P(x(1), r_1, v) \ldots P(x(k), r_k, v) \ldots$ approach the solution value, V_o, of the convex programming problem as $r_k \to 0$ ($k \to \infty$) i.e.,

$$\lim_{\substack{r_k \to 0 \\ k \to \infty}} \left[\min_{x \in R} P[x, r_k, v] \right] = V_o .$$

Proof:

Let $\epsilon > 0$ be chosen. The consider $\bar{\chi}$ such that $\bar{\chi} \in R^{\circ}$, and $f(\bar{\chi}) < V_0 + \frac{\epsilon}{2}$

Select r_l from $r_1 \cdots r_l \cdots$ such that

$$r_l \left[\max_i \left(\frac{1}{g_i(\bar{\chi})} \right)^{\nu} \right] < \frac{\epsilon}{2m} .$$

Then for

$$r_p < r_l \quad (p > l)$$

$\min_{\chi \in R} P(x, r_p, v)$ exists (Theorem 1)

$$= P[\chi(p), r_p, \nu] \qquad \text{(Lemma 4)}$$

$$< P[\chi(q), r_q, \nu] \qquad \text{(Lemma 5)}$$

$$\leq P[\bar{\chi}, r_q, \nu]$$

$$= f(\bar{\chi}) + r_q \sum_i \left(\frac{1}{g_i(\bar{\chi})} \right)^{\nu}$$

$$< V_0 + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= V_0 + \epsilon$$

Since $P[\chi(p), r_p, \nu] > f[\chi(p)] \geq V_0 > V_0 - \epsilon$

then

$$V_0 - \epsilon < P[\chi(p), r_p, \nu] < V_0 + \epsilon$$

This completes the proof of the convergence of P(x,r,v) to the minimum of the convex programming problem.

Lemma 6.

a) $$\lim_{\substack{r_p \to 0 \\ p \to \infty}} r_p \sum_i \left(\frac{1}{g_i[\chi(p)]} \right)^{\nu} = 0$$

b) $$\lim_{\substack{r_p \to 0 \\ p \to \infty}} f[\chi(p)] = V_0$$

Proof: The following inequalities are true.

(1) $$f[\chi(k+1)] + r_{k+1} \sum_i \left(\frac{1}{g_i[\chi(k+1)]}\right)^r \le f[\chi(k)] + r_{k+1} \sum_i \left(\frac{1}{g_i[\chi(k)]}\right)^r$$

and (2) $$f[\chi(k)] + r_k \sum_i \left(\frac{1}{g_i[\chi(k)]}\right)^r \le f[\chi(k+1)] + r_k \sum_i \left(\frac{1}{g_i[\chi(k+1)]}\right)^r$$

Adding and transposing: $$(r_{k+1} - r_k) \sum_i \left(\frac{1}{g_i[\chi(k+1)]}\right)^r \le (r_{k+1} - r_k) \sum_i \left(\frac{1}{g_i[\chi(k)]}\right)^r$$

Since $(r_{k+1} - r_k) < 0$ then

(3) $$\sum_i \left(\frac{1}{g_i[\chi(k+1)]}\right)^r \ge \sum_i \left(\frac{1}{g_i[\chi(k)]}\right)^r$$

from (1), $$f[\chi(k+1)] - f[\chi(k)] \le r_{k+1} \left(\sum_i \left(\frac{1}{g_i[\chi(k)]}\right)^r - \sum_i \left(\frac{1}{g_i[\chi(k+1)]}\right)^r \right)$$

From (3) the right hand side is negative which implies;

(4) $$f[\chi(k)] \ge f[\chi(k+1)]$$

is a decreasing sequence bounded below, (condition C6), by V_0 .

Therefore, $$\lim_{k \to \infty} f[\chi(k)] \ge V_0$$

and

$$\lim_{\substack{r_k \to 0 \\ k \to \infty}} [P[\chi(k), r_k, r] - f[\chi(k)]] = \lim_{\substack{k \to \infty \\ r_k \to 0}} r_k \sum_i \left(\frac{1}{g_i[\chi(k)]}\right)^r$$

$$V_0 - \lim_{k \to \infty} f[\chi(k)] \ge 0$$

since

$$r_k \sum_i \left(\frac{1}{g_i[\chi(k)]}\right)^r \ge 0$$

which implies;

$$\lim_{\substack{k \to \infty \\ r_k \to 0}} f[\chi(k)] = V_0$$

therefore,

$$\lim_{\substack{r_k \to 0 \\ k \to \infty}} r_k \sum \left(\frac{1}{g_i[x(k)]} \right)^v = 0 \ .$$

This proves Lemma 6.

For convenience we restate the dual of the convex programming problem:

Maximize $\quad G[x, \lambda] = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$

Subject to: $\quad \nabla_x f(x) = \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x)$

$$\lambda_i \geq 0$$

Theorem 3: Under conditions C1 thru C6 the method yields points

$$(x(k), \lambda(k)) \ , \quad \lambda_i = v r_k \left(\frac{1}{g_i[x(k)]} \right)^{v+1}$$

that are dual feasible, and values of $G[x(k), \lambda(k)]$ with,

$$\lim_{\substack{r_k \to 0 \\ k \to \infty}} G[x(k), \lambda(k)] = V_0$$

Proof:

From theorem 1.b, the gradient of P(x(k), r_k ,v) is equal to zero, that is:

$$\nabla_x P[x(k), r_k, v] = \nabla_x f[x(k)] - v r_k \sum_i \nabla_x g_i[x(k)] \left(\frac{1}{g_i[x(k)]} \right)^{v+1} = 0$$

which implies:

$$\nabla_x f[x(k)] = v r_k \sum_i \nabla_x g_i[x(k)] \left(\frac{1}{g_i[x(k)]} \right)^{v+1} .$$

Choosing $$\lambda_i(k) = v\, r_k \left(\frac{1}{g_i[x(k)]}\right)^{v+1}.$$

Then $$\nabla_x f[x(k)] = \sum_i \lambda_i \nabla_x g_i[x(k)],$$

Thus, at any P minimum, the dual side constraints are satisfied by

$$[x(k), \lambda(k)].$$

From Lemma 6, $\lim f[x(k)] = V_0$ as $r_k \to 0$ and $\lim r_k \sum_i \left(1/g_i[x(k)]\right)^v = 0$ then for an $\epsilon > 0$ an $r_{k(\epsilon)}$ can be found such that for $r_k < r_{k(\epsilon)}$

$$V_0 \leq f[x(k)] < V_0 + \epsilon$$

and $$-\epsilon < -r_k \sum_i \left(\frac{1}{g_i[x(k)]}\right)^v < 0$$

Adding yields $$V_0 - \epsilon < f[x(k)] - r_k \sum_i \left(\frac{1}{g_i[x(k)]}\right)^v < V_0 + \epsilon$$

Thus, $$G[x(k), \lambda(k)] = f[x(k)] - r_k \sum_i \left(\frac{1}{g_i[x(k)]}\right)^v$$
$$\leq V_0 \leq P[x(k), r_k, v]$$

Therefore for any r_k minimum the solution V_o to the convex programming problem is bounded above, by P(x(k), r_k , v), and below, by $G[x(k), \lambda(k)]$. This result provides a criterion for termination of the method.

The only requirement on v, for the proof of convergence of the P(x, r, v) function, is v > 0. The P(x, r) function of SUMT is equal to the P(x, r, v) function for v = 1, therefore the unconstrained function of SUMT is a special case of the P(x, r, v) function.

5. Behavior of the Modified SUMT as a function of v.

Up to now very little has been said as to what the introduction of the new variable v would accomplish. The original idea behind the development of the P(x, r, v) function was to increase the rate of convergence of SUMT for problems with solution on the boundary. Will $v > 0$ and not equal to one increase the rate of convergence of SUMT? If so, what is the optimum value of v? The answers to those questions are quite complicated for the general case. The behavior of the modified SUMT for a simple function will be described below.

The following example demonstrates the effect of v on the P function minima.

Minimize $x_1 + x_2$

Subject to $x_1 \geq 0$

$x_2 \geq 0$

The unconstrained problem is

$$\text{Minimize } P(x, r, v) = x_1 + x_2 + r\left(\frac{1}{x_1}\right)^v + r\left(\frac{1}{x_2}\right)^v$$

By theorem 1, the gradient of P is the null vector at a minimum, for a fixed value of $r > 0$ and $v > 0$. This implies

$$x_1 = \left(rv\right)^{v+1} = x_2$$

The minimum occurs at (0,0) with $V_o = 0$. Table I lists the minimum points and values of P(x, r, v) for a fixed value of v and a reduction of r at each step. It is obvious from Table I, that the smallest value of v, i.e., v = .125, produces the largest reduction in the P function and values of x_1 and x_2 that are closest to the desired value.

The SUMT computing program[9] was modified for the P(x, r, v) function. The sample program above was used for a test of the procedure.

TABLE 1

VALUES OF THE P FUNCTION FOR DIFFERENT VALUES OF r AND v.

	r	x_1	x_2	P(x, r, v)
v = 3	1.	1.31	1.31	3.5
	.5	1.09	1.09	2.91
	.25	.93	.93	2.47
	.125	.788	.788	2.078
	.0625	.65	.65	1.74
v = 2	1.	1.26	1.26	3.78
	.5	1.	1.	3.0
	.25	.705	.705	2.21
	.125	.63	.63	1.86
	.0625	.5	.5	1.5
v = 1	1.	1.	1.	4.0
	.5	.71	.71	2.83
	.25	.5	.5	2.0
	.125	.35	.35	1.40
	.0625	.25	.25	1.0
v = .75	1.	.92	.92	3.94
	.5	.57	.57	2.64
	.25	.48	.48	1.96
	.125	.25	.25	1.20
	.0625	.172	.172	.804
v = .5	1.	.64	.64	3.80
	.5	.4	.4	2.4
	.25	.25	.25	1.5
	.125	.16	.16	.94
	.0625	.10	.10	.6

	r	x_1	x_2	P(x,r,v)
v = .25	1.	.33	.33	3.30
	.5	.19	.19	1.61
	.25	.108	.108	1.086
	.125	.06	.06	.62
	.0625	.035	.035	.37
v = .125	1.	.19	.19	2.90
	.5	.085	.085	1.53
	.25	.045	.045	.82
	.125	.025	.025	.45
	.0625	.013	.013	.226

Table II tabulates the results achieved. The solution of the sample problem using SUMT is also shown. The time difference between the modified version of SUMT with v = 1 and the unmodified SUMT is due to the additional computation required for the exponent v. As expected, the time required for solution reduces as v is reduced. If SUMT is used as a base for comparisons then v $<$ .75 is required (for this problem) to produce a reduction in computational time. Although this is a simple problem it serves to illustrate the reduction of computer time expected for a convex programming problem, whose solution lies on the boundary of one or more constraints.

If the value of v is too small (close to zero) the penalty term tends to a constant, that is,

$$\lim_{v \to 0} r_k \sum_{i=1}^{m} \left(\frac{1}{g_i(x)} \right)^v = r_k m$$

Thus v must be greater than zero. The effect of v, (in the interval (0, 1)), on the penalty function varies with each constraint. For those constraints whose values are greater than one, (reciprocal less than one), v increases their values and effect on the penalty function. For those constraints whose values are less than one, (reciprocal greater than one), v decreases their values and effect on the penalty function. Thus the effect of v on the penalty term, the gradient of P, and the matrix of second partials is directly related to the values of the constraints.

SUMT, uses a second order gradient method[5,8] to minimize P(x, r, v), given by

$$x^2 = x' - \alpha \left\| \frac{\partial^2 P[x', r, v]}{\partial x_i \partial x_j} \right\|^{-1} \nabla_x P[x', r, v]$$

Where α is determined by a search procedure to minimize the function P(x, r, v) along the modified gradient. Then the process is repeated, starting from the point x^2, until a minimum of the P(x, r, v) is achieved.

TABLE 2

MODIFIED SUMT SOLUTIONS OF THE SAMPLE PROBLEM

v	x_1	x_2	Minimum P(x,r,v)	Computer time in seconds
3.	19.E-05*	16.E-05	43.E-05	44.066
2.	12.E-05	15.E-05	37.E-05	34.433
1.	41.E-06	76.E-06	18.E-05	24.76
.75	57.E-06	27.E-06	14.E-05	22.75
.5	54.E-06	23.E-06	15.E-05	18.666
.25	84.E-07	11.E-06	88.E-06	16.217
.125	44.E-07	46.E-07	80.E-06	14.434
.0625	21.E-07	68.E-07	75.E-06	13.584
.02	34.E-08	54.E-07	85.E-06	12.650
.01	24.E-08	38.E-08	71.E-06	12.467
.005	16.E-08	16.E-08	66.E-06	12.484
.002	30.E-08	59.E-09	63.E-06	12.966
.001	33.E-09	17.E-09	62.E-06	12.350
SUMT solution (unmodified)				
	43.E-06	43.E-06	17.E-05	20.05

*Note: 19.E-05 = $19. \times 10^{-5}$ = .00019

The criteria used to terminate the procedure is that the magnitude of the gradient be less than some pre-assigned small positive number, i.e.,

$$\left| \nabla_x P[x, r, v] \right| < \epsilon$$

The parameter $v < 1$ affects the gradient by the factor v and by the exponent $v + 1$ which is less than the exponent 2 of SUMT. Also the matrix of second partials is affected by the factor $v(v + 1)$ and the exponent $v + 2$ which is less than the exponent 3 of SUMT, (Section 4).

Since it is almost impossible to determine a priori whether the solution point is interior or on the boundary of the feasible region, the parameter v, in the interval (0,1), should also reduce the computational time for problems with interior solutions.

As explained above, v, in the interval (0,1), reduces the effect of the constraint on the penalty function and on the second order gradient method. For a problem with an interior point solution, this is a desirable feature. If we knew ahead of time that the solution was an interior point, then we could eliminate the constraints and minimize f(x). Therefore $v < 1$ should serve to accelerate the rate of convergence of convex programming problems with interior solutions.

6. Conclusions and Acknowledgements.

Fiacco, McCormack, and Mylander[11] demonstrated that the SUMT program converges to the solution of a convex programming problem, and that their method is as efficient as any other method in current use. The proof of convergence of the P(x, r, v) function for $v > 0$ is shown in Section 4. For v in the interval (0,1) the rate of convergence is accelerated for a sample problem. This increase in the rate of convergence is not expected to be as pronounced for all classes of nonlinear problems; however, it is expected that $0 < v < 1$ will reduce the computational time of SUMT for nonlinear problems with solutions on one or more constraint boundaries. And it is expected that the introduction of v will not increase the time required for a problem with an interior solution.

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The Sequential Unconstrained Minimization Technique (SUMT) for Convex Programming Problems is modified by the introduction of an exponent in the penalty term. The exponent is introduced to increase the rate of convergence of the method for nonlinear problems with solutions on the boundary of one or more constraints. Convergence to the solution of the constrained problem is proved and it is shown that SUMT is a special case of the general unconstrained function with the exponent equal to one. Results of a sample problem indicate that the rate of convergence is improved and that the computational time for solution is decreased for an exponent less than one.

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